Let the events $A_1, A_2, \ldots, A_n$ represent a partition of the sample space $S$. Let $B$ be any other event defined on $S$. If $P(A_i) \neq 0$, $i = 1, 2, \ldots, n$ and $P(B) \neq 0$ then

$$P(A_i | B) = \frac{P(A_i) \times P(B | A_i)}{\sum P(A_i) \times P(B | A_i)}$$

If we write $p_1 = P(A_1)$, $p_2 = P(A_2)$, $p_3 = P(A_3)$, etc.

and $p'_1 = P(B | A_1)$, $p'_2 = P(B | A_2)$, $p'_3 = P(B | A_3)$, etc.

then Bayes theorem can be stated as

$$P(A_i | B) = \frac{P_i p'_i}{P_1 p'_1 + P_2 p'_2 + \ldots + P_n p'_n}$$

The characteristic function of a random variable $X$ denoted by $\Phi_X(\omega)$ is defined by

$$\Phi_X(\omega) = E(e^{i\omega X})$$

where $\omega$ is an auxiliary variable.

$$\Phi_X(\omega) = \sum e^{i\omega x} p(x)$$

(For discrete probability distribution)

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f(x) dx$$

(For continuous probability distribution)

If $F_X(x)$ is the distribution function of a continuous random variable then

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} dF(x)$$

Since, $|\Phi(\omega)| = \left| \int_{-\infty}^{\infty} e^{i\omega x} f(x) dx \right| = \int_{-\infty}^{\infty} |e^{i\omega x}| f(x) dx$

But $|e^{i\omega x}| = \sqrt{\cos^2 \omega x + \sin^2 \omega x} = 1$

$\therefore |\Phi(\omega)| = \int |f(x)| dx = 1$ (Since $f(x)$ is a density function)

Since $|\Phi(\omega)| \leq 1$, $\Phi(\omega)$ always exists.

Properties of Autocorrelation function $R(\tau)$

$R(\tau)$ is an even function $\tau$.

Proof: $R(\tau) = E[X(t) \cdot X(t+\tau)]$ \quad $R[-\tau] = E[X(t) \cdot X(t-\tau)]$

Putting $t - \tau = t'$ i.e. $t = t' + \tau$

$R[-\tau] = E[X(t'+\tau) X(t')]$

$= E[X(t') \cdot X(t' + \tau)]$

$= R(\tau)$

When we say $X_n = i$, it means the process is in the $i$-th state at time $n$. It is assumed that all the relevant probabilities are known. Thus, we assume that we know the probability $p_{ij}$ i.e. the probability that the process will go from $i$-th state to the $j$-th state, for all $i$, $j$ and $n \geq 0$. If this transition is independent of the past i.e. if the transition that the process will go from $i$-th state to $j$-th state depends only on the present state $i$, the process is called Markov Chain.

Example: Consider an experiment of tossing a fair coin for a number of times. At each toss there are two possibilities either head with probability $1/2$ or tail with probability $1/2$. 


Let us consider each toss as a Bernoulli’s random variable with
\[ P(X = 0) = \frac{1}{2} \]
\[ P(X = 1) = \frac{1}{2} \]
where, \( X \) denotes the number of heads obtained.

If we toss the coin repeatedly we have \( X_1, X_2, \ldots X_n \ldots \) Bernoulli’s random variables.

Consider now a random variable \( S_n \) defined by \( S_n = X_1 + X_2 + \ldots + X_n \).
\( S_n \) is the random variable which is the sum of \( n \) independent and identical random variables. \( S_n = 0 \) if all \( X_i \) are zero. \( S_n = 1 \) if one \( X_i \) is 1 and all the remaining \( X_i \) are zero. \( S_n = 2 \), if two \( X_i \) are 1 and the remaining are zero and so on. For any \( n \), \( S_n \) takes any value between 0 and \( n \).

Suppose \( S_n = k \) where \( k \) is between 0 and \( n \). Then the random variable \( S_n \) for \( n+1 \) will take the value \( k+1 \) or \( k \) each with probability 1/2
\[ P(S_{n+1} = k+1 / S_n = k) = \frac{1}{2} \]
\[ P(S_{n+1} = k / S_n = k) = \frac{1}{2} \]

Thus, the probability that \( S_{n+1} \) will take value \( k+1 \) or will remain at \( k \) does not depend upon the previous states but depends upon the state \( S_n = k \). The value of \( S_{n+1} \) given \( S_n = k \) depends upon the value of \( S_n \) and not on how \( S_n \) was reached.

Suppose we have tossed a fair coin 40 times and in these 40 tosses we obtained 16 heads. Then in the 41-st toss we may get head with probability 1/2 or we may get tail with probability 1/2.

Thus, on 41\(^{st} \) trail the number of heads will go from 16 to 17 with probability 1/2 or will remain at 16 with probability 1/2.
\[ P(S_{40+1} = 16 + 1/S_{40} = 16) = \frac{1}{2} \]
\[ P(S_{40+1} = 16 / S_{40} = 16) = \frac{1}{2} \]

This is a Markov Chain. The future does not depend upon the past but depends only on the present.

2. (a) Since \( \Theta \) is uniformly distributed on \((0, 2\pi)\)
\[ f_\Theta(\theta) = \frac{1}{2\pi}, \quad 0 < \theta < 2\pi \]
\[ \therefore \quad E[X(t)] = E[A \cos (\omega_0 t + \Theta)] \]
\[ \therefore \quad E[X(t)] = \int_0^{2\pi} A \cos (\omega_0 t + \theta) \cdot \frac{1}{2\pi} \cos (\omega_0 t + \theta) d\theta \]
\[ = \frac{A}{2\pi} \left[ \sin (\omega_0 t + 2\pi) - \sin (\omega_0 t) \right] = \frac{A}{2\pi} \sin (\omega_0 t - \sin (\omega_0 t) \]
\[ = 0, \quad \text{a constant.} \]

Further, \( R_{X,X}(\tau) = E[X(t) \cdot X(t+\tau)] = E[A \cos(\omega_0 t+\Theta) \cdot A \cos(\omega_0 (t+\tau)+\Theta)] \]
\[ = \frac{A^2}{2} \cdot E[\cos(2\omega_0 t + \omega_0 \tau + 2\theta) + \cos(\omega_0 \tau)] \]
\[ R_{X, X}(\tau) = \frac{A^2}{2} \left[ \int_0^{2\pi} \frac{1}{2\pi} \cos(2\omega_0 t + \omega_0 \tau + 2\theta) d\theta + \int_0^{2\pi} \frac{1}{2\pi} \cos(\omega_0 \tau) d\theta \right] \]
\[ = \frac{A^2}{2} \left[ 2 \sin(2\omega_0 t + \omega_0 \tau + 2\theta) \right]_0^{2\pi} + \frac{1}{2\pi} \cos(\omega_0 \tau) \cdot \left[ \theta \right]_0^{2\pi} \]
\[ = \frac{A^2}{2} \left[ 0 + \frac{1}{2\pi} \cos(\omega_0 \tau)[2\pi - 0] \right] \]
\[ = \frac{A^2}{2} \cos \omega_0 \tau \]

From (1) and (2), it follows that \( X(t) \) is WSS.
2. (b) In the first place there are two possibilities either urn A is selected or urn B is selected.

(i) **Urn A is selected**: Probability of selecting urn A is 1/2 and getting an even number is 1/2.

Now a ticket is drawn from urn B. Probability of getting an even number from B is 4/9.

Probability of thus getting two even numbers = \( \frac{1}{2} \cdot \frac{4}{9} = \frac{2}{9} \).

**Urn B is selected**: Probability of selecting urn B is 1/2 and getting an even number is 4/9.

Now, ticket is drawn from urn A. Probability of getting an even number from A is 1/2.

Probability of thus getting two even numbers = \( \frac{1}{2} \cdot \frac{4}{9} = \frac{2}{9} \).

Required probability = \( \frac{2}{9} + \frac{2}{9} = \frac{4}{9} \).

(ii) **Urn A is selected**: Probability of selecting urn A is 1/2 and getting an odd number is 1/2.

Now, another ticket is drawn from A. Probability of getting an odd number again is 3/7.

Probability of thus getting two odd numbers = \( \frac{1}{2} \cdot \frac{3}{7} = \frac{3}{14} \).

**Urn B is selected**: Probability of selecting urn B is 1/2 and getting an odd number is 5/9.

Now, another ticket is drawn from urn A. Probability of getting an odd number is 4/8 = 1/2.

Probability of thus getting two odd numbers = \( \frac{1}{2} \cdot \frac{4}{8} = \frac{1}{4} \).

Required Probability = \( \frac{3}{14} + \frac{1}{4} = \frac{5}{28} \).

3. (a) (i) The marginal probability density of X is given by

\[
f_X(x) = f_{X,Y}(x,y) \mid_{y=-\infty}^{y=\infty} = k_1k_2 e^{-(k_1x + k_2y)} dy\bigg|_{y=-\infty}^{y=\infty} = k_1k_2 e^{-k_2y} dy\bigg|_{y=0}^{y=\infty} = k_1k_2 e^{-k_2y} \bigg|_{y=0}^{y=\infty} = k_1e^{-k_2x}, \quad 0 < x < \infty
\]

(ii) Similarly, the marginal probability density of y is \( f_Y(y) = k_2e^{-k_2y}, \quad 0 < y < \infty \)

(iii) Distribution function of (X, Y) is

\[
F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v) du \, dv = \int_{0}^{x} \int_{0}^{y} k_1k_2 e^{-(k_1u + k_2v)} du \, dv
\]

\[
= k_1 \int_{0}^{x} 0^{-k_1u} du \cdot k_2 \int_{0}^{y} e^{-k_2v} dv
\]

\[
= k_1 \left[ -\frac{e^{-k_1u}}{k_1} \right]_{0}^{x} \cdot k_2 \left[ -\frac{e^{-k_2v}}{k_2} \right]_{0}^{y}
\]

\[
= \left(1-e^{-k_1x}\right) \left(1-e^{-k_2y}\right), \quad 0 < x < \infty, 0 < y < \infty
\]
(iv) \[ P(X > Y) = \int_0^\infty \int_0^\infty f_{XY}(x, y) \, dx \, dy = \int_0^\infty \int_0^\infty k_1 k_2 e^{-(k_1 x + k_2 y)} \, dx \, dy \]
\[ = \int_0^\infty k_1 e^{-k_1 x} \left[ \int_0^\infty k_2 e^{-k_2 y} \, dy \right] \, dx \]
\[ = \int_0^\infty k_1 e^{-k_1 x} \left[ e^{k_2 y} \right]_0^\infty \, dx = \int_0^\infty k_1 e^{-k_1 x} \left( 1 - e^{-k_2 x} \right) \, dx \]
\[ = \int_0^\infty k_1 e^{-k_1 x} \, dx - \int_0^\infty k_1 e^{-(k_1 + k_2) x} \, dx \]
\[ = \left[ -e^{-k_1 x} \right]_0^\infty - k_1 \left[ -\frac{e^{-(k_1 + k_2) x}}{k_1 + k_2} \right]_0^\infty \]
\[ = [1 - 0] - \frac{k_1}{k_1 + k_2} [1 - 0] = \frac{k_2}{k_1 + k_2}. \]

3. (b) Properties of Power Spectral Density Function

1) The value of the spectral density function at zero frequency is equal to the total area under the autocorrelation function.

**Proof:** Putting \( f = 0 \) in (4) \([\text{or } \omega = 0 \text{ in (2)}]\), we get

\[ S(0) = \int_{-\infty}^{\infty} R(\tau) \, d\tau \]

The l.h.s. is the value of the spectral density function at zero frequency and the r.h.s. is the area under the autocorrelation function. Hence, the result.

2) The mean square value of a wide-sense stationary process is equal to the area under spectral density function.

**Proof:** We know that \( R(\tau) = E[X(t) \cdot X(t + \tau)] \)
Putting \( \tau = 0 \), we get \( R(0) = E[X^2(t)] \)
But by (8),
\[ R(\tau) = \int_{-\infty}^{\infty} S_\lambda(f) e^{j2\pi ft} \, df \]
Putting \( \tau = 0 \), we get
\[ R(0) = \int_{-\infty}^{\infty} S_\lambda(f) \, df \]
From (16) and (17), we get \( E[X^2(t)] = \int_{-\infty}^{\infty} S_\lambda(f) \, df \) which is the required result.

3) The spectral density function of a real random process is an even function.

**Proof:** By definition (2) the spectral density function
\[ S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{j\omega \tau} \, d\tau \]
Changing the sign of \( \omega \), \( S(-\omega) = \int_{-\infty}^{\infty} R(\tau) e^{j\omega \tau} \, d\tau \]
Putting \( \tau = -u \), \( d\tau = -du \), when \( \tau \rightarrow -\infty \), \( u \rightarrow \infty \) and when \( \tau \rightarrow \infty \), \( u \rightarrow -\infty \)
\[ S(-\omega) = \int_{-\infty}^{\infty} R(-u)e^{-i\omega u} (-du) = \int_{-\infty}^{\infty} R(-u)e^{-i\omega u} du \]

But \( R(\tau) \) is an even function of \( \tau \), \( \therefore R(-\tau) = R(\tau) \)

\[ S(-\omega) = \int_{-\infty}^{\infty} R(\tau)e^{i\omega \tau} d\tau = S(\omega) \]

Hence, \( S(\omega) \) is an even function.

4) The spectral density function of a random process \( X(t) \) which may be real or complex, is a real function.

**Proof** : By definition

\[ R(\tau) = E[X(t) \cdot X^*(t + \tau)] \quad \text{where} \quad X^* \text{ denotes the complex conjugate of } X. \]

\[ \therefore R(-\tau) = E[X(t) \cdot X^*(t-\tau)] \]

Taking the conjugates,

\[ \therefore R^*(\tau) = E[X^*(t) \cdot X(t-\tau)] = R(\tau) \]

(Changing the sign of \( \tau \))

\[ i.e. \ R^*(\tau) = R(-\tau) \]

Now, \( S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega \tau} d\tau \)

\[ \therefore S^*(\omega) = \int_{-\infty}^{\infty} R^*(\tau)e^{i\omega \tau} d\tau = \int_{-\infty}^{\infty} R(-\tau)e^{i\omega \tau} d\tau \quad \text{[By (1)]} \]

Putting \( u = -\tau \) as above, we get

\[ S^*(\omega) = \int_{-\infty}^{\infty} R(u)e^{-i\omega u} du = S(\omega) \]

Hence, \( S(\omega) \) is a real function.

5) The spectral density and the autocorrelation function of a real WSS form a Fourier cosine transform pair.

**Proof** : By definition

\[ S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega \tau} d\tau = \int_{-\infty}^{\infty} R(\tau)[\cos \omega \tau - i \sin \omega \tau] d\tau \]

Since, the first integral is even and the second is odd

\[ S(\omega) = 2 \int_{0}^{\infty} R(\tau) \cos \omega \tau d\tau = \text{Fourier cosine transform of } [2 R(\tau)] \]

(For Fourier transform see the appendix.)

Again \( R(\tau) = \frac{1}{2\pi} \int_{0}^{2\pi} S(\omega)e^{i\omega \tau} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)(\cos \omega \tau + \sin \omega \tau) d\omega \)

As before the first integral is even and the second is odd.

\[ \therefore R(\tau) = \frac{1}{\pi} \int_{0}^{\infty} S(\omega) \cos \omega \tau d\omega = \text{Fourier inverse cosine transform of } \left[ \frac{1}{2} S(\omega) \right] \]

6) **White Noise**

**Definition** : A random process \( \omega(t) \) whose power spectral density \( S(\omega) \) is constant at all frequencies.

\[ i.e. \quad S_\omega(\omega) = \frac{N_0}{2} \quad \text{where} \quad N_0 \text{ is a real positive constant is called white noise.} \]
Since white light also contains all frequencies in equal amounts, this process is called white noise. The autocorrelation function is given by

$$R_\omega(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{N_0}{2} \right) e^{-i\omega \tau} d\tau = \frac{N_0}{2} \cdot \frac{1}{2\pi} \cdot 2\pi \delta(\tau)$$

$$\vdash R_\omega(\tau) = \frac{N_0}{2} \cdot \delta(\tau)$$

The average power is given by

$$P_\omega(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{N_0}{2} \right) d\omega$$

Since this is infinite, white noise is unrealisable.

The two functions $R_\omega(\tau) = \frac{N_0}{2} \cdot \delta(\tau)$ and $S_\omega(\omega) = \frac{N_0}{2}$ are shown in the following figures.

4. (a) (i) $P(X$ less than 1 inch$) = \int_{0}^{\frac{3}{4}} (2x - x^2) dx = \frac{3}{4} \left[ x^3 - \frac{x^4}{3} \right]_0 = \frac{3 \cdot 2}{4} = \frac{1}{2}$

(ii) $P(X$ greater than 1.5 inches$) = \int_{1.5}^{\frac{3}{2}} (2x - x^2) dx$

$$= \frac{3}{4} \left[ x^3 - \frac{x^4}{3} \right]_{1.5}^{\frac{3}{2}} = \frac{3}{4} \left[ 4 - \frac{9}{8} \right] = \frac{3}{4} \cdot \frac{5}{24} = \frac{5}{32}$$

(iii) $P(X$ lies between 0.5 and 1.5$) = \int_{0.5}^{1.5} (2x - x^2) dx$

$$= \frac{3}{4} \left[ x^3 - \frac{x^4}{3} \right]_{0.5}^{1.5} = \frac{3}{4} \left[ \left( \frac{9}{4} - \frac{9}{8} \right) - \left( \frac{1}{4} \right) \right]$$

$$= \frac{3}{4} \left[ \frac{22}{4} \right] = \frac{22}{16} = \frac{11}{8}$$

4. (b) By definition, $R_X(\tau) = F^{-1} [Z_X(\omega)]$

$$R_X(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega \tau}}{(1 + \omega^2)^2} d\omega \quad ....(1)$$
By considering the contour integration consisting of a large semi-circle with centre \( t \) the origin, in the upper half of the plane and its diameter on the real axis (Refer to Applied Mathematics IV by the same author).

\[
\int_{-\infty}^{\infty} \frac{e^{iaz}}{(1 + x^2)^2} \, dx = 2\pi i \text{(Sum of the residues lying in the upper half)}
\]

Now, the singularity in the upper half is the double pole \( z = i \).

\[
(\text{Residue})_{z=i} = \frac{1}{1!} \left[ \frac{d}{dz} \left( \frac{e^{iaz}}{(z+i)^2} \right) \right]_{z=i}
\]

\[
= \left( \frac{(z+i)^2 \cdot e^{iaz} - e^{ia} \cdot 2(z+i)}{(z+i)^4} \right)_{z=i}
\]

\[
= \frac{-4e^{-a} \cdot i - 4ie^{-a}}{16} = -\frac{e^{-a} \cdot i + (1+a)}{4}
\]

\[
\int_{-\infty}^{\infty} \frac{e^{iaz}}{(1 + x^2)^2} \, dx = 2\pi i \left( \frac{-e^{-a} \cdot i + (1+a)}{4} \right)
\]

From (1) and (2), we get

\[
R_X(\tau) = \frac{1}{4} (1 + \tau) e^{-\tau}
\]

But the average power is given by \( R_X(0) \).

Hence, average power \( P_{XX}(t) = \frac{1}{4} = 0.25 \)

5. (a) Taking the Fourier transform of \( R(\tau) \), the power spectral density \( S(\omega) \) is given by

\[
S(\omega) = \int_{-\infty}^{\infty} R_X(\tau) \cdot e^{-i\omega \tau} \, d\tau
\]

\[
= \int_{-\infty}^{\infty} e^{-\tau^2} \cos \omega_0 \tau \cdot e^{-i\omega \tau} \, d\tau
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} e^{-\tau^2} \cdot e^{-i(\omega-\omega_0)\tau} \, d\tau + \frac{1}{2} \int_{-\infty}^{\infty} e^{-\tau^2} \cdot e^{-i(\omega+\omega_0)\tau} \, d\tau
\]

Now, consider

\[
\int_{-\infty}^{\infty} e^{-\tau^2} \cdot e^{-i(\omega-\omega_0)\tau} \, d\tau
\]

By completing the square on \( \tau \),

\[
\int_{-\infty}^{\infty} e^{-\left(\tau + \frac{i(\omega-\omega_0)}{2a}\right)^2} \cdot \frac{(\omega-\omega_0)^2}{4a} \, d\tau = e^{-\frac{(\omega-\omega_0)^2}{4a}} \int_{-\infty}^{\infty} e^{-\left(\tau + \frac{i(\omega-\omega_0)}{2a}\right)^2} \, d\tau
\]

Putting \( \tau + \frac{i(\omega-\omega_0)^2}{2} = x \), \( d\tau = dx \)

\[
I = \int_{-\infty}^{\infty} e^{-ax^2} \, dx = \sqrt{\frac{\pi}{a}} \]

\[
\therefore \int_{-\infty}^{\infty} e^{-\tau^2} \cdot e^{-i(\omega-\omega_0)\tau} \, d\tau = e^{-\frac{(\omega-\omega_0)^2}{4a}} \cdot \sqrt{\frac{\pi}{a}}
\]

\[
I = \int_{-\infty}^{\infty} e^{-ax^2} \, dx = \sqrt{\frac{\pi}{a}} \int_{-\infty}^{\infty} e^{-\frac{(\omega-\omega_0)^2}{4a}} \, d\tau
\]

\[
\therefore \int_{-\infty}^{\infty} e^{-\tau^2} \cdot e^{-i(\omega-\omega_0)\tau} \, d\tau = e^{-\frac{(\omega-\omega_0)^2}{4a}} \cdot \sqrt{\frac{\pi}{a}}
\]
Similarly changing the sign of $\omega_0$

\[
\int_{-\infty}^{\infty} e^{-at^2} \cdot e^{-i(\omega+\omega_0)t} \, dt = \frac{e^{-\frac{(\omega-\omega_0)^2}{4a}}}{\sqrt{\pi a}} \cdot \frac{\pi}{\sqrt{a}}
\]

Hence,

\[
S(\omega) = \frac{1}{2} \left[ \frac{e^{-\frac{(\omega-\omega_0)^2}{4a}}}{\sqrt{\pi a}} + \frac{e^{-\frac{(\omega+\omega_0)^2}{4a}}}{\sqrt{\pi a}} \right]
\]

5. (b) Let $U = XY$ and $V = Y$

\[
\therefore \ u = xy \text{ and } v = y \quad \therefore \ x = \frac{u}{v} \text{ and } y = v
\]

\[
J = \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{vmatrix} = \begin{vmatrix}
\frac{1}{v} & \frac{u}{v^2} \\
0 & 1
\end{vmatrix} = \frac{1}{u}
\]

Since $X, Y$ are independent random variates

\[
f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}
\]

\[
\therefore \ f_{U,V}(u, v) = J |f_{X,Y}(x, y)| = \frac{1}{u} \cdot \frac{1}{6} = \frac{1}{6u}
\]

Now, when $x = 1$,

\[
\therefore \ x = \frac{u}{v}, \ u = v \text{ and when } x = 2, \quad \therefore \ x = \frac{u}{v}, \ u = 2v
\]

When $y = 2$,

\[
\therefore \ y = v, \ v = 2 \quad \text{and when } y = 4, \quad \therefore \ y = v, \ v = 4
\]

Thus, $f_{U,V}(u, v) = \frac{1}{6}$ and the region is bounded by $v = 2, v = 4, u = v$ and $u = 2v$.

We obtain $f_{u}(u)$ i.e. the marginal probability density function of $u$ by integrating out $u$.

In the region I, $v$ varies $v = 2$ to $v = u$.

\[
\therefore \ f_{u}(u) = \frac{1}{6} \int_{v=2}^{u} dv = \frac{1}{6} [v]_{2}^{u} = \frac{1}{6} (u - 2)
\]

In the region II, $v$ varies $v = \frac{u}{2}$ to $v = 4$.

\[
\therefore \ f_{u}(u) = \int_{u/2}^{4} \frac{1}{6} dv = \frac{1}{6} [v]_{u/2}^{4} = \frac{1}{6} [4 - \frac{u}{2}] = \frac{8 - u}{12}
\]

\[
\therefore \ f_{u}(u) = \begin{cases} 
\frac{u - 2}{6}, & 2 < u < 4 \\
\frac{8 - u}{12}, & 4 < u < 8
\end{cases}
\]
6. (a) We are given that

\[ P(T_0) = a \text{ 0 is transmitted} = 0.45 \]
\[ P(T_1) = a \text{ 1 is transmitted} = 1 - P(T_0) = 1 - 0.45 = 0.55 \]

\[ P(R_0 / T_0) = a \text{ 0 is received when a 0 was transmitted} = 0.9 \]
\[ P(R_1 / T_0) = a \text{ 1 received when a 0 was transmitted} = 1 - 0.9 = 0.1 \]

\[ P(R_0 / T_1) = a \text{ 0 is received when a 1 was transmitted} = 0.2 \]
\[ P(R_1 / T_1) = a \text{ 1 is received when a 1 was transmitted} = 0.8 \]

Now, we calculate the required probabilities as follows:

(i) \[ P(1 \text{ is received}) = P(1 \text{ is received when 1 is transmitted}) + P(1 \text{ is received when 0 is transmitted}) \]
\[ \therefore P(R_1) = P(R_1 / T_1) \cdot P(T_1) + P(R_1 / T_0) \cdot P(T_0) = 0.8 \times 0.55 + 0.1 \times 0.45 = 0.485 \]

(ii) \[ P(0 \text{ is received}) = P(0 \text{ is received when 0 is transmitted}) + P(0 \text{ is received when 1 is transmitted}) \]
\[ \therefore P(R_0) = P(R_0 / T_0) \cdot P(T_0) + P(R_0 / T_1) \cdot P(T_1) = 0.9 \times 0.45 + 0.2 \times 0.55 = 0.515 \]

Now, by Bayes’ Theorem

(iii) \[ P(1 \text{ was transmitted given that 1 was received}) = \frac{P(T_1 / R_1)}{P(R_1)} \]
\[ = \frac{1}{0.907} = 0.907 \]

(iv) \[ P(0 \text{ was transmitted given that 1 was received}) = \frac{P(T_0 / R_0)}{P(R_0)} \]
\[ = \frac{0.515}{0.786} = 0.659 \]

(v) \[ P(\text{Error}) = P(\text{0 was received when 1 is transmitted given that 1 was transmitted}) + P(\text{1 was received when 0 was transmitted given that 0 was transmitted}) \]
\[ = P(R_0 / T_1) \cdot P(T_1) + P(R_1 / T_0) \cdot P(T_0) = 0.2 \times 0.55 + 0.1 \times 0.45 = 0.155 \]

6. (b) Definition: A random variable is said to follow the Binomial distribution if the probability of \( x \) is given by

\[ P(X = x) = ^nC_x p^x q^{n-x}, x = 0, 1, 2, 3, ..., n \text{ and } q = 1 - p \]

Mean and Variance: The first two moments about the origin are obtained as follows.

\[ \mu_1 = E(X) = \sum_{x=0}^{n} nx \cdot p^x q^{n-x} \cdot x = nC_0 p^0 q^n + nC_1 pq^ {n-1} \cdot 1 + nC_2 p^2 q^{n-2} \cdot 2 + .... + p^n \cdot n \]
\[ = npq^{n-1} + \frac{n(n-1)}{2!} p^2 q^{n-2} \cdot 2 + .... + p^n \cdot n \]
\[ = np \left[ q^{n-1} + (n-1)q^{n-2} \cdot p + \frac{(n-1)(n-2)}{2!} q^{n-3} p^3 + ... + p^{n-1} \right] \]
\[ \therefore \mu_1 = np[q + p]^{n-1} = np \quad [\because p + q = 1] \]
\[ \mu_2' = E(X^2) = \Sigma p \cdot x^2 = \Sigma x C_x p^x q^{n-x} \]

But \( x^2 \) can be written as \( x^2 = x + x(x-1) \)

\[ \therefore \mu_2' = \Sigma [x + x(x-1)] C_x p^x q^{n-x} \]

\[ = \sum x^n C_x p^x q^{n-x} + \sum x(x-1) n C_x p^x q^{n-x} \]

But the first term on the r.h.s. is \( np \) as shown above.

\[ \therefore \mu_2 = \text{np} + \left[ 0^n C_0 p^0 q^n + 0^n C_1 p^1 q^{n-1} + 2 \cdot 1^n C_2 p^2 q^{n-2} + 3 \cdot 2^n C_3 p^3 q^{n-3} + \ldots \right] \]

\[ = \text{np} + 2 \cdot \frac{n(n-1)}{2!} p^2 q^{n-2} + 3 \cdot \frac{n(n-1)(n-2)}{3!} p^3 q^{n-3} + 4 \cdot \frac{n(n-1)(n-2)(n-3)}{4!} p^4 q^{n-4} + \ldots \]

\[ = \text{np} + n(n-1) \pi^2 q^{n-2} + n(n-1)(n-2) p^2 q^{n-3} + \ldots \]

\[ = \text{np} + n(n-1) \pi^2 q^{n-2} + \left( n(n-1)(n-2) p^2 q^{n-3} + \ldots \right) \] \[ \quad \text{[∵ p + q = 1]} \]

\[ = \text{np} \left[ 1 + (n-1)p \right] = \text{np} \left[ 1 + np \right] \]

\[ = \text{np}^2 + np^2 \]

\[ \therefore \mu_2 = \mu_2' - \mu_1^2 = npq \]

Mean = \( np \) and Variance = \( npq \)

7. (a) \( X_2 = 3 \) means the state 2 and the condition is 3.

We have \( P^2 = \begin{bmatrix} 0.2 & 0.5 & 0.3 \\ 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.2 & 0.5 & 0.3 \\ 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.38 & 0.42 & 0.20 \\ 0.33 & 0.45 & 0.22 \\ 0.35 & 0.43 & 0.22 \end{bmatrix} \)

(i) \( P(X_2 = 3) = \text{[Initial distribution]} \times \text{[third column of } P^2\text{]} \)

\[ = \begin{bmatrix} 0.6 & 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} 0.20 \\ 0.22 \end{bmatrix} \]

\[ = 0.6 \times 0.2 + 0.3 \times 0.22 + 0.1 \times 0.22 = 0.12 + 0.066 + 0.022 = 0.208 \]

(ii) \( P(X_2 = 2) = \text{[Initial Distribution]} \times \text{[Second column of } P^2\text{]} \)

\[ = \begin{bmatrix} 0.6 & 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} 0.42 \\ 0.45 \end{bmatrix} \]

\[ = 0.6 \times 0.42 + 0.3 \times 0.45 + 0.1 \times 0.43 = 0.252 + 0.135 + 0.043 = 0.43. \]

(iii) Since \( P(A/B) = \frac{P(A \cap B)}{P(B)} \)

\[ \therefore P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2) \]

\[ = P[X_3 = 2 \mid X_2 = 3, X_1 = 3, X_0 = 2] \times P[X_2 = 3, X_1 = 3, X_0 = 2] \]

\[ = P[X_3 = 2 \mid X_2 = 3, X_1 = 3, X_0 = 2] \times P[X_2 = 3, X_1 = 3, X_0 = 2] \]

\[ = P[X_3 = 2 \mid X_2 = 3] \times P[X_2 = 3, X_1 = 3, X_0 = 2] \]

\[ \quad \ldots (1) \]

(By Markovian property the probability of \( X_3 = 2 \) depends upon the probability of previous state \( X_2 = 3 \) only. It has no concern with \( X_1 = 3 \) and \( X_2 = 2 \))
Now, \[ P[X_2 = 3, X_1 = 3, X_0 = 2] = P[X_2 = 3 | X_1 = 3, X_0 = 2] \cdot P[X_1 = 3, X_0 = 2] \]
\[ = \left( \frac{P[X_2 = 3 | X_1 = 3]}{P[X_1 = 3, X_0 = 2]} \right) \cdot P[X_1 = 3, X_0 = 2] \quad \cdots (2) \]
(By Markovian property, the probability of \( X_2 = 3 \) depends upon the probability of the previous state \( X_1 = 3 \) only. It has no concern with \( X_0 = 2 \).)

And \[ P[X_1 = 3, X_0 = 2] = P[X_1 = 3 | X_0 = 2] \cdot P[X_0 = 2] \quad \cdots (3) \]

Hence, from (3), we get \[ P[X_1 = 3, X_0 = 2] = 0.1 \times 0.3 = 0.03 \quad \cdots (4) \]

From (2), we get \[ P[X_2 = 3, X_1 = 3, X_0 = 2] = 0.3 \times 0.03 = 0.009 \quad \text{[By (4)]} \quad \cdots (5) \]

From (1), we get \[ P[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2] = 0.4 \times 0.009 = 0.0036 \quad \text{[By (5)]} \]

7. (b) A Poisson process is a stochastic process in which events occur continuously and independently of one another (the word event used here is not an instance of the concept of event frequently used in probability theory). Examples that are well-modeled as Poisson processes include the radioactive decay of atoms, telephone calls arriving at a switchboard, page view requests to a website, and rainfall. A Poisson process is usually described as a function of time, although it need not be.

The Poisson process is a collection \( \{N(t), t \geq 0\} \) of random variables, where \( N(t) \) is the number of events that have occurred up to time \( t \) (starting from time 0). The number of events between time \( a \) and time \( b \) is given as \( N(b) - N(a) \) and has a Poisson distribution. Each realization of the process \( \{N(t)\} \) is a non-negative integer-valued step function that is non-decreasing, but for intuitive purposes it is usually easier to think of it as a point pattern on \([0, \infty)\) (the points in time where the step function jumps, i.e. the points in time where an event occurs).

The Poisson Process is Markov Process

**Proof:** Consider

\[ P[X(t_3) = n_3 | X(t_2) = n_2, X(t_1) = n_1] \]
\[ = \frac{P[X(t_3) = n_3, X(t_2) = n_2, X(t_1) = n_1]}{P[X(t_2) = n_2, X(t_1) = n_1]} \quad \text{[Using the above two results.]}
\]
\[ = P[X(t_3) = n_3 | X(t_2) = n_2] \quad \text{[By (5)]} \]

Thus, the conditional probability of \( X(t_3) = n_3 \) depends only on \( X(t_2) = n_2 \) and not on \( X(t_1) \) i.e. on the distant past.

In other words the Poisson process satisfies the Markovian property. Hence, the Poisson process is a Markov process.